

ON THE PERSISTENCE PROPERTIES OF SOLUTIONS OF NONLINEAR DISPERSIVE EQUATIONS IN WEIGHTED SOBOLEV SPACES

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ABSTRACT. We study persistence properties of solutions to some canonical dispersive models, namely the semi-linear Schrödinger equation, the k -generalized Korteweg-de Vries equation and the Benjamin-Ono equation, in weighted Sobolev spaces $H^s(\mathbb{R}^n) \cap L^2(|x|^l dx)$, $s, l > 0$.

1. INTRODUCTION

This work is concerned with persistence properties of solutions to some nonlinear dispersive equations in weighted Sobolev spaces $H^s(\mathbb{R}^n) \cap L^2(|x|^l dx)$, $s, l > 0$. We shall consider the initial value problems (IVP) associated to the following dispersive models : the nonlinear Schrödinger (NLS) equation

$$(1.1) \quad i\partial_t u + \Delta u = \mu |u|^{a-1} u, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad \mu = \pm 1, \quad a > 1,$$

the k -generalized Korteweg-de Vries (k -gKdV) equations

$$(1.2) \quad \partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \quad t, x \in \mathbb{R}, \quad k \in \mathbb{Z}^+,$$

and the Benjamin-Ono (BO) equation

$$(1.3) \quad \partial_t u + \mathcal{H} \partial_x^2 u + u \partial_x u = 0, \quad t, x \in \mathbb{R},$$

where \mathcal{H} denotes the Hilbert transform

$$(1.4) \quad \mathcal{H}f(x) = \frac{1}{\pi} \lim_{\epsilon \downarrow 0} \int_{|y| \geq \epsilon} \frac{f(x-y)}{y} dy = -i (\operatorname{sgn}(\xi) \hat{f}(\xi))^\vee(x).$$

These models have been widely studied in several contexts. For example, the KdV $k = 1$ in (1.2) was first deduced as a model for long waves propagating in a channel. Subsequently the KdV and its modified form ($k = 2$ in (1.2)) were found to be relevant in a number of different physical systems. Also they have been studied because of their relation to inverse scattering theory [20]. The NLS arises as a model in several different physical phenomena (see [58] and references therein). In the particular, case $n = 1$ and $a = 3$ it has been shown to be completely integrable [63]. The BO equation (1.3) was first deduced in [3] and [51] as a model for long internal gravity waves in deep stratified fluids. It was also shown that it is a completely integrable system (see [2], [12] and references therein).

We recall the notion of well posedness given in [33] : the IVP is said to be locally well posed (LWP) in the function space X if for each $u_0 \in X$ there exist $T > 0$ and a unique solution $u \in C([-T, T] : X) \cap \dots = Y_T$ of the equation, with the map data \rightarrow solution being locally continuous from X to Y_T . This notion of LWP includes the “persistent” property, i.e. the solution describes a continuous curve on X . In particular, it implies that the solution flow defines a dynamical system in X . When

T can be taken arbitrarily large one says that the corresponding IVP is globally well posed (GWP) in X .

First, we shall study the Schrödinger equation (1.1).

2. THE SCHRÖDINGER EQUATION (1.1)

The results in [9], [10], [21], [34], and [62] yield the following LWP theory in the classical Sobolev spaces $H^s(\mathbb{R}^n)$ for the IVP associated to the NLS equation (1.1).

Theorem A. *Let $s_c = n/2 - 2/(a-1)$.*

- (I) *If $s > s_c$, $s \geq 0$, with $[s] \leq a-1$ if a is not an odd integer, then for each $u_0 \in H^s(\mathbb{R}^n)$ there exist $T = T(\|u_0\|_{s,2}) > 0$ and a unique solution $u = u(x, t)$ of the IVP associated to the NLS equation (1.1) with*

$$(2.1) \quad u \in C([-T, T] : H^s(\mathbb{R}^n)) \cap L^q([-T, T] : L_s^p(\mathbb{R}^n)) = Z_T^s.$$

Moreover, the map data \rightarrow solution is locally continuous from $H^s(\mathbb{R}^n)$ into Z_T^s .

- (II) *If $s = s_c$ and $s \geq 0$, then part (I) holds with $T = T(u_0) > 0$.*

Notations : (a) for $1 < p < \infty$ and $s \in \mathbb{R}$

$$(2.2) \quad L_s^p(\mathbb{R}^n) \equiv (1 - \Delta)^{-s/2} L^p(\mathbb{R}^n) = J^{-s/2} L^p(\mathbb{R}^n), \quad \|\cdot\|_{s,p} \equiv \|(1 - \Delta)^s \cdot\|_p,$$

with $L_s^2(\mathbb{R}^n) = H^s(\mathbb{R}^n)$,

(b) the pair of indices (q, p) in (2.1) are given by the Strichartz estimates (see [57] and [21]):

$$(2.3) \quad \left(\int_{-\infty}^{\infty} \|e^{it\Delta} u_0\|_p^q dt \right)^{1/q} \leq c \|u_0\|_2,$$

where

$$\frac{n}{2} = \frac{2}{q} + \frac{n}{p}, \quad 2 \leq p \leq \infty, \quad \text{if } n = 1, \quad 2 \leq p < 2n/(n-2), \quad \text{if } n \geq 2.$$

The value $s_c = n/2 - 2/(a-1)$ in Theorem A is determined by a scaling argument : if $u(x, t)$ is a solution of the IVP associated to the NLS equation (1.1), then $u_\lambda(x, t) = \lambda^{2/(a-1)} u(\lambda x, \lambda^2 t)$ satisfies the same equation with data $u_\lambda(x, 0) = \lambda^{2/(a-1)} u_0(\lambda x)$. Hence, for $s \in \mathbb{R}$

$$(2.4) \quad \|D^s u_\lambda(x, 0)\|_2 = c \|\xi|^s \widehat{u_\lambda}(\xi, 0)\|_2 = c \lambda^{2/(a-1) + s - n/2} \|u_0\|_2,$$

is independent of λ when $s = s_c$. In Theorem A the case (I) corresponds to the sub-critical case and (II) to the critical one. In the latter, one has that if $\|D^{s_c} u_0\|_2$ is sufficiently small, then the local solution extends globally in time.

For the optimality of the results in Theorem A see [4], [11], and [39].

Formally, solutions of the NLS equation (1.1) satisfies the following conservation laws:

$$\|u(\cdot, t)\|_2 = \|u_0\|_2,$$

and

$$E(t) = \int_{\mathbb{R}^n} (|\nabla_x u(x, t)|^2 + \frac{2\mu}{a+1} |u(x, t)|^{a+1}) dx = E(0).$$

Using these conservation laws one can extend the LWP results in Theorem A to a GWP one, for details we refer to [6], [61], and references therein.

Concerning the persistence properties in weighted Sobolev spaces of solutions of the IVP associated to the NLS equation (1.1) one has the following result established in [25], [26], and [27].

Theorem B. *In addition to the hypothesis in Theorem A assume $u_0 \in L^2(|x|^{2m} dx)$, $m \in \mathbb{Z}^+$ with $m \leq a - 1$ if a is not an odd integer.*

(I) *If $s \geq m$, then*

$$(2.5) \quad u \in C([-T, T] : H^s \cap L^2(|x|^{2m} dx)) \cap L^q([-T, T] : L^p_s \cap L^p(|x|^{2m} dx)) = Z_T^{s,m}.$$

(II) *If $1 \leq s < m$, then (2.5) holds with $[s]$ instead of m and*

$$(2.6) \quad \Gamma^\beta u = (x_j + 2it\partial_{x_j})^\beta u \in C([-T, T] : L^2) \cap L^q([-T, T] : L^p),$$

for any $\beta \in (\mathbb{Z}^+)^n$ with $|\beta| \leq m$.

The proof of Theorem B (see [25], [26], [27]) combines the operators (“vector fields”)

$$(2.7) \quad \Gamma_j = x_j + 2it\partial_{x_j} = e^{i|x|^2/4t} 2it\partial_{x_j} (e^{-i|x|^2/4t} \cdot) = e^{it\Delta} x_j e^{-it\Delta}, \quad j = 1, \dots, n,$$

their commutative relation

$$(2.8) \quad (i\partial_t + \Delta)\Gamma_j u = \Gamma_j(i\partial_t u + \Delta u), \quad j = 1, \dots, n,$$

so that $e^{it\Delta}(x_j u_0) = \Gamma_j e^{it\Delta} u_0$, and the structure of the nonlinearity in (1.1).

It should be remarked that Theorem B shows that the amount of decay in $L^2(|x|^{2m} dx)$ preserved by the solution depends on the regularity in the Sobolev scale H^s , $s \geq 0$) of the data, and the non-preserved decay is transformed in “local regularity”. In particular, (2.6) tells us that $t^\beta \partial_x^\beta u \in L^2_{loc}(\mathbb{R}^n)$, for $|\beta| \leq m$ and $t \in [-T, T] - \{0\}$.

Also one notices that the power of the weight m in Theorem B is assumed to be an integer. In [50] we were able to remove this restriction.

Theorem 1. *In addition to the hypothesis in Theorem A assume $u_0 \in L^2(|x|^{2m} dx)$, $m > 0$ with $[m] \leq a - 1$ if a is not an odd integer.*

(I) *If $s \geq m$,*

$$(2.9) \quad u \in C([-T, T] : H^s \cap L^2(|x|^{2m} dx)) \cap L^q([-T, T] : L^p_s \cap L^p(|x|^{2m} dx)) = Z_T^{s,m}.$$

(II) *If $1 \leq s < m$, then (2.9) holds with $[s]$ instead of m and*

$$(2.10) \quad \Gamma^b \Gamma^\beta u(\cdot, t) \in C([-T, T] : L^2) \cap L^q([-T, T] : L^p),$$

where $\Gamma^b = e^{i|x|^2/4t} 2^b t^b D^b (e^{-i|x|^2/4t} \cdot)$ with $|\beta| = [m]$ and $b = m - [m]$.

In particular,

$$(2.11) \quad t^m \partial_x^\beta D^b u(\cdot, t) \in L^2_{loc}(\mathbb{R}^n), \quad |\beta| = [m], \quad b = m - [m], \quad t \in (-T, T) - \{0\}.$$

As an application of this result we also prove that the persistence property in these weighted spaces can only hold for regular enough solutions. More precisely:

Lemma 1. *Let u be a solution of the IVP associated to the NLS equation (1.1) provided by Theorem A. If there exist two times $t_1, t_2 \in [0, T]$, $t_1 \neq t_2$ such that*

$$(2.12) \quad |x|^m u(t_1), \quad |x|^m u(t_2) \in L^2(\mathbb{R}^n), \quad m > s,$$

$m \leq a - 1$ if a is not an odd integer, then

$$u \in C([-T, T] : H^m \cap L^2(|x|^{2m} dx)) \cap L^q([-T, T] : L_m^p \cap L^p(|x|^{2m} dx)).$$

Moreover, if a is an odd integer and (2.12) holds for all $m \in \mathbb{Z}^+$, then

$$(2.13) \quad u \in C([-T, T] : \mathbb{S}(\mathbb{R}^n)).$$

A key ingredient in our proof was an appropriate version of the Leibnitz rule for homogeneous fractional derivatives of order $b \in \mathbb{R}$

$$(2.14) \quad D^b f(x) \equiv ((2\pi|\xi|)^b \hat{f})^\vee(x)$$

deduced as a direct consequence of the characterization of the $L_s^p(\mathbb{R}^n)$ spaces (see (2.2)) given in [55].

Theorem D. *Let $b \in (0, 1)$ and $2n/(n + 2b) \leq p < \infty$. Then $f \in L_b^p(\mathbb{R}^n)$ if and only if*

$$(a) \quad f \in L^p(\mathbb{R}^n),$$

(2.15)

$$(b) \quad \mathcal{D}^b f(x) = \left(\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x - y|^{n+2b}} dy \right)^{1/2} \in L^p(\mathbb{R}^n),$$

with

$$(2.16) \quad \|f\|_{b,p} = \|(1 - \Delta)^{b/2} f\|_p \simeq \|f\|_p + \|D^b f\|_p \simeq \|f\|_p + \|\mathcal{D}^b f\|_p.$$

For the proof of Theorem D we refer to [55], where the optimality of the lower bound $2n/(n + 2b)$ was also established. The case $p = 2n/(n + 2b)$ was proven in [18]. For a detailed discussion on the different characterizations of the $L_s^p(\mathbb{R}^n)$ spaces we refer to [55] and [56].

It is easy to see that for $p = 2$ and $b \in (0, 1)$ one has

$$(2.17) \quad \|\mathcal{D}^b f\|_2 \simeq \|D^b f\|_2,$$

$$(2.18) \quad \|\mathcal{D}^b(fg)\|_2 \leq c(\|f\mathcal{D}^b g\|_2 + \|g\mathcal{D}^b f\|_2),$$

and for $p > 2n/(n + 2b)$

$$(2.19) \quad \mathcal{D}^b(fg)(x) \leq \|f\|_\infty \mathcal{D}^b g(x) + |g(x)| \mathcal{D}^b f(x).$$

We observe that in (2.18) both terms on the right hand side are estimates on the product of functions. We do not know whether or not (2.18) still holds with D^b instead of \mathcal{D}^b , or for $p \neq 2$.

Theorem D (i.e. the estimates (2.18)-(2.17)) allows us to get the following inequalities:

–(i) Let $b \in (0, 1)$. For any $t > 0$

$$(2.20) \quad \mathcal{D}^b(e^{it|x|^2}) \leq c(t^{b/2} + t^b|x|^b).$$

–(ii) Let $b \in (0, 1)$. Then there exists $c = c(b) > 0$ such that for any $t \in \mathbb{R}$

$$(2.21) \quad \| |x|^b e^{it\Delta} f \|_2 \leq c(t^{b/2} \|f\|_2 + t^b \|D^b f\|_2 + \| |x|^b f \|_2).$$

–(iii) Defining the operator Γ^b for $b > 0$ as in Theorem 1 (see (2.10))

$$(2.22) \quad \Gamma^b \equiv \Gamma^b(t) = e^{i|x|^2/4t} 2^b t^b D^b \left(e^{-i|x|^2/4t} \cdot \right),$$

one has for $b > 0$ and $t \in \mathbb{R}$ that

$$(2.23) \quad \Gamma^b(t)e^{it\Delta}f = e^{it\Delta}(|x|^b f),$$

and consequently

$$(2.24) \quad \Gamma^b(t)f = e^{it\Delta}(|x|^b e^{-it\Delta}f).$$

In addition to the estimates (2.20)-(2.24) the following two lemmas were essential in the proof of Theorem 1 given in [50]. The first is a version of the Gagliardo-Nirenberg inequality for fractional derivatives.

Lemma 2. *Let $1 < q, p, r < \infty$ and $0 < \alpha < \beta$. Then*

$$(2.25) \quad \|D^\alpha f\|_p \leq c \|f\|_r^{1-\theta} \|D^\beta f\|_q^\theta,$$

with

$$(2.26) \quad \frac{1}{p} - \frac{\alpha}{n} = (1-\theta)\frac{1}{r} + \theta\left(\frac{1}{q} - \frac{\beta}{n}\right), \quad \theta \in [\alpha/\beta, 1].$$

The second is an interpolation estimate, which as Lemma 2, is a consequence of the three line theorem.

Lemma 3. *Let $a, b > 0$. Assume that $J^a f = (1 - \Delta)^{a/2} f \in L^2(\mathbb{R})$ and $\langle x \rangle^b f = (1 + |x|^2)^{b/2} f \in L^2(\mathbb{R})$. Then for any $\theta \in (0, 1)$*

$$(2.27) \quad \|J^{\theta a}(\langle x \rangle^{(1-\theta)b} f)\|_2 \leq c \|\langle x \rangle^b f\|_2^{1-\theta} \|J^a f\|_2^\theta.$$

For the study of persistence properties of the solution to the IVP associated to the NLS equation (1.1) in exponential weighted spaces we refer to [16], [17], and references therein.

Next, we shall consider the k -gKdV equation (1.2).

3. THE k -GENERALIZED KORTEWEG-DE VRIES EQUATION (1.2)

The following theorem describes the LWP theory in the classical Sobolev spaces $H^s(\mathbb{R})$ for the IVP associated to the kg KdV equation (1.2).

- Theorem E.** (I) *The IVP associated to the equation (1.2) with $k = 1$ is LWP in $H^s(\mathbb{R})$ for $s \geq s_1^* = -3/4$.*
 (II) *The IVP associated to the equation (1.2) with $k = 2$ is LWP in $H^s(\mathbb{R})$ for $s \geq s_2^* = 1/4$.*
 (III) *The IVP associated to the equation (1.2) with $k = 3$ is LWP in $H^s(\mathbb{R})$ for $s \geq s_3^* = -1/6$.*
 (IV) *The IVP associated to the equation (1.2) with $k \geq 4$ is LWP in $H^s(\mathbb{R})$ for $s \geq s_k^* = (k-4)/2k$.*

The result $s > -3/4$ for the case $k = 1$ was established in [38]. The limiting value $s = -3/4$ was obtained in [11]. The result for the case $k = 2$ was proven in [37]. The result $s > -1/6$ for the case $k = 3$ was given in [22]. The limiting value $s = -1/6$ was obtained in [60]. The proof of the cases $k \geq 4$ was given in [37].

The above local results apply to both real and complex valued functions.

The scaling argument described in (2.4) affirms that LWP should hold for $s \geq s_k = (k-4)/2k$. As Theorem E shows this is the case for $k \geq 3$ (where for $s_k = s_k^*$ one has $T = T(u_0)$). However, in the cases $k = 1$ and $k = 2$ the values suggested

by the scaling do not seem to be reachable in the Sobolev scale, see [39], and [11]. For the sharpness of these results we refer to [4], [39], and [11].

Real valued solutions of the k -gKdV equation (1.2) formally satisfy at least three conservation laws:

$$\begin{aligned} I_1(u) &= \int_{-\infty}^{\infty} u(x, t) dx, & I_2(u) &= \int_{-\infty}^{\infty} (u(x, t))^2 dx, \\ I_3(u) &= \int_{-\infty}^{\infty} ((\partial_x u(x, t))^2 - \frac{2}{(k+1)(k+2)} u(x, t)^{k+2}) dx. \end{aligned}$$

It was proven in [13] that for $k = 1$ and $k = 2$ one has global well posedness for $s > -3/4$ and $s > 1/4$, respectively. The global cases for $k = 1$, $s = -3/4$ and $k = 2$, $s = 1/4$ were proven in [24] and [41]. For the case $k = 3$ the global well posedness is known for $s > -1/42$, see [23].

For $k = 4$ blow up of “large” enough solutions was proven in [45]. Similar results for $k \geq 5$ remain an open problem.

Concerning the persistence of these solutions in weighted Sobolev spaces one has the following result found in [33].

Theorem F. *Let $m \in \mathbb{Z}^+$. Let $u \in C([-T, T] : H^s(\mathbb{R})) \cap \dots$ with $s \geq 2m$ be the solution of the IVP associated to the equation (1.2) provided by Theorem E. If $u(x, 0) = u_0(x) \in L^2(|x|^{2m} dx)$, then*

$$u \in C([-T, T] : H^s(\mathbb{R}) \cap L^2(|x|^{2m} dx)).$$

We recall that if for a solution $u \in C([0, T] : H^s(\mathbb{R}))$ of (1.2) one has that $\exists t_0 \in [0, T]$ such that $u(\cdot, t_0) \in H^{s'}(\mathbb{R})$, $s' > s$, then $u \in C([0, T] : H^{s'}(\mathbb{R}))$. So we shall mainly consider the most interesting case $s = 2m$ in Theorem F.

The proof of Theorem F combines the operator

$$\Gamma = x + 3t\partial_x^2,$$

and its commutative relation with the linear part $L = \partial_t + \partial_x^3$ of the equation (1.2) i.e.

$$\Gamma(\partial_t + \partial_x^3)v = (\partial_t + \partial_x^3)\Gamma v.$$

As in the case of the NLS equation (1.1) we would like to extend Theorem F where $m \in \mathbb{Z}^+$ to the case $m \in \mathbb{R}$, $m > 0$. Our first result in this direction is the following:

Theorem 2. *Let $m \geq 0$. Let $u \in C([-T, T] : H^m(\mathbb{R})) \cap \dots$ with $m \geq \max\{s_k^*; 0\}$ be the solution of the IVP associated to the equation (1.2) provided by Theorem E. If $u(x, 0) = u_0(x) \in L^2(|x|^m dx)$, then*

(I) *If $m < 1$, then for any $\epsilon > 0$*

$$u \in C([-T, T] : H^m(\mathbb{R}) \cap L^2(|x|^{m-\epsilon} dx)).$$

(II) *If $m \geq 1$, then*

$$u \in C([-T, T] : H^m(\mathbb{R}) \cap L^2(|x|^m dx)).$$

In [48] and [49] the loss of power $\epsilon > 0$ in the weight when $m < 1$ was removed for the equation (1.2) with non-linearity $k = 2, 4, 5, \dots$. More precisely, the following optimal result was established in [49]:

Theorem 3. *Let $m \geq \max\{s_k^*; 0\}$ with $k = 2, 4, 5, \dots$. Let $u \in C([-T, T] : H^m(\mathbb{R})) \cap \dots$ be the solution of the IVP associated to the equation (1.2) provided by Theorem E. If $u(x, 0) = u_0(x) \in L^2(|x|^m dx)$, then*

$$u \in C([-T, T] : H^m(\mathbb{R}) \cap L^2(|x|^m dx)).$$

It should be remarked that in the cases $k = 1$ and $k = 3$ the proof of the local theory in Theorem E is based on the spaces $X_{s,b}$ introduced in the context of dispersive equations in [5]. For all the other powers k one has a local existence theory based on a contraction principle in a spaces defined by mixed norms of the type $L^p(\mathbb{R} : L^q([0, T]))$ or $L^q([0, T] : L^p(\mathbb{R}))$ (see [37]). This is the main difficulty in extending the optimal result in Theorem 3 to the powers $k = 1$ and $k = 3$ in (1.2).

Proof of Theorem 2

We shall sketch the ideas in the proof of Theorem 2 and refer to [48] and [49] for the justification of the argument and further details.

Following Kato's idea in [33] to establish the local smoothing effect (i.e. multiplying the equation (1.2) by $u(x, t)\phi(x)$, integrating the result, and using integration by parts) one formally gets the identity

$$(3.1) \quad \frac{d}{dt} \int u^2 \phi dx + 3 \int (\partial_x u)^2 \phi' dx - \int u^2 \phi^{(3)} dx - \frac{2}{k+2} \int u^{k+2} \phi' dx = 0.$$

Let us consider first the case $\max\{s_k^*; 0\} \leq m < 1$.

From the local theory one has the following estimates for the solution $u = u(x, t)$

$$(3.2) \quad \sup_{x \in \mathbb{R}} \left(\int_0^T |\partial_x D_x^m u(x, t)|^2 dt \right)^{1/2} < c_T \|J^m u_0\|_2 = c_T \|u_0\|_{m,2},$$

(the sharp form of the local smoothing effect found in [36]-[37]), and

$$(3.3) \quad \begin{aligned} \|D_x^m u\|_{L_x^2 L_T^2} &= \left(\int_{-\infty}^{\infty} \int_0^T |D_x^m u(x, t)|^2 dt dx \right)^{1/2} \\ &\leq T^{1/2} \sup_{t \in [0, T]} \|D_x^m u(t)\|_2 < c_T \|D_x^m u_0\|_2 \leq c_T \|u_0\|_{m,2}. \end{aligned}$$

Now, we consider the extensions of the estimates in (3.2)-(3.3) to the operators D_x^{1+m+iy} and D_x^{m+iy} , $y \in \mathbb{R}$ respectively. First, in the linear case one has the estimates

$$(3.4) \quad \begin{aligned} \|D_x^{m+1+iy} v\|_{L_x^\infty L_T^2} &\leq c_T \|D_x^m v_0\|_2, \\ \|D_x^{m+iy} v\|_{L_x^2 L_T^2} &\leq c_T \|D_x^m v_0\|_2, \end{aligned}$$

for

$$(3.5) \quad v(x, t) = U(t)v_0(x) = c \int_{-\infty}^{\infty} e^{ix\xi} e^{it\xi^3} \widehat{v}_0(\xi) d\xi.$$

To apply the three line theorem we consider the function $F(z)$ defined on $\mathcal{S} = \{z \in \mathbb{C} : \Re(z) \in [0, 1]\}$

$$F(z) = \int_{-\infty}^{\infty} \int_0^T D_x^{s(z)} v(x, t) \phi(x, z) f(t) dt dx,$$

where

$$s(z) = (1 - z)(1 + m) + zm, \quad 1/q(z) = (1 - z) + z/2, \quad q = 2/(2 - m),$$

$$\phi(x, z) = |g(x)|^{q/q(z)} \frac{g(x)}{|g(x)|}, \quad \text{with} \quad \|g\|_{L_x^{2/(2-m)}} = \|f\|_{L^2([0, T])} = 1,$$

which is analytic on the interior of \mathcal{S} . So using that

$$\|\phi(\cdot, 0 + iy)\|_1 = \|\phi(\cdot, 1 + iy)\|_2 = 1,$$

one gets that

$$(3.6) \quad \begin{aligned} \|\partial_x v\|_{L_x^{2/m} L_T^2} &\leq c \|D_x v\|_{L_x^{2/m} L_T^2} \\ &\leq c \sup_{y \in \mathbb{R}} \|D_x^{1+m+iy} v\|_{L_x^\infty L_T^2}^{1-m} \sup_{y \in \mathbb{R}} \|D_x^{m+iy} v\|_{L_x^2 L_T^2}^m \leq c_T \|D^m v_0\|_2. \end{aligned}$$

Inserting the estimate (3.6) in the proof of the local well posedness one obtains that

$$(3.7) \quad \|\partial_x u\|_{L_x^{2/m} L_T^2} \leq c_T \|u_0\|_{m,2},$$

for $u = u(x, t)$ solution of the k -gKdV equation (1.2).

Now taking $\phi(x) = \langle x \rangle^{m-\epsilon}$, $\epsilon > 0$ sufficiently small in (3.1), (we recall that $m < 1$) and integrating in the time interval $[0, T]$ one finds that

$$(3.8) \quad \begin{aligned} \int_0^T \int_{-\infty}^{\infty} (\partial_x u(x, t))^2 \phi'(x) dx dt &= c \|\partial_x u \langle x \rangle^{\frac{m}{2} - \frac{1}{2} - \frac{\epsilon}{2}}\|_{L_x^2 L_T^2}^2 \\ &\leq c \|\langle x \rangle^{m/2-1/2-\epsilon/2}\|_{L_x^{2/(1-m)}} \|\partial_x u\|_{L_x^{2/m} L_T^2} \leq c_{m,\epsilon} \|\partial_x u\|_{L_x^{2/m} L_T^2}, \end{aligned}$$

which combined with (3.6) and (3.1) shows that $\langle x \rangle^{m/2-\epsilon/2} u(\cdot, t) \in L^2(\mathbb{R})$ for $t \in [0, T]$. This basically completes the proof of the case $m < 1$.

Next, we shall consider the case $m \geq 1$.

We take in (3.1) $\phi(x) = \langle x \rangle^m$ in (3.1), so we need to estimate the term

$$\int_{-\infty}^{\infty} |\partial_x u(x, t)|^2 \langle x \rangle^{m-1} dx = \|\partial_x u(\cdot, t) \langle \cdot \rangle^{(m-1)/2}\|_{L_x^2}^2.$$

Thus, combining Lemma 3 in the previous section, the preservation of the L^2 -norm of the solution, and Lemma 3 it follows that

$$(3.9) \quad \begin{aligned} &\|\partial_x u(\cdot, t) \langle \cdot \rangle^{(m-1)/2}\|_2 \\ &\leq \|\partial_x (u(\cdot, t) \langle \cdot \rangle^{(m-1)/2})\|_2 + c \|u(\cdot, t) \langle \cdot \rangle^{(m-3)/2}\|_2 \\ &\leq \|\partial_x J^{-1} J(u(\cdot, t) \langle \cdot \rangle^{(m-1)/2})\|_2 + c \|u(\cdot, t) \langle \cdot \rangle^{m/2}\|_2 \\ &\leq c \|J(u(\cdot, t) \langle \cdot \rangle^{(m-1)/2})\|_2 + c \|u(\cdot, t) \langle \cdot \rangle^{m/2}\|_2 \\ &\leq c \|J^m u(\cdot, t)\|_2^{1/m} \|u(\cdot, t) \langle \cdot \rangle^{m/2}\|_2^{1-1/m} + c \|u(\cdot, t) \langle \cdot \rangle^{m/2}\|_2. \end{aligned}$$

Hence, inserting (3.9) in (3.1), using Young and Gronwall inequalities, the hypothesis $m \geq 1$, and the fact that the H^m -norm of the solution is bounded in the time interval $[0, T]$ one obtains the desired result

$$\sup_{t \in [0, T]} \|\langle x \rangle^{m/2} u(\cdot, t)\|_{L^2} < \infty.$$

This completes the sketch of the proof of Theorem 2.

To finish this section concerning the k -gKdV equation (1.2) we will make some comments concerning the proof of Theorem 3 given in [48] and [49]. One of the key element in that proof is the following commutator estimate:

Lemma 4. *Let $0 < \alpha < 1$ and $1 < p < \infty$. Then for functions $f, g : \mathbb{R} \rightarrow \mathbb{C}$ one has that*

$$(3.10) \quad \|D^\alpha(fg) - fD^\alpha g\|_p \leq c \|Q_N(D^\alpha f)\|_{L^\infty l_N^1} \|g\|_2,$$

where

$$\|Q_N(f)\|_{L^\infty l_N^1} \equiv \left\| \sum_{N \in \mathbb{Z}} |Q_N(f)| \right\|_{L^\infty},$$

and

$$Q_N(f)(x) = ((\eta\left(\frac{\xi}{2^N}\right) + \eta\left(-\frac{\xi}{2^N}\right))\widehat{f}(\xi))^\vee(x),$$

where $\eta \in C_0^\infty(\mathbb{R})$ with $\text{supp}(\eta) \subseteq [1, 2, 2]$ so that

$$\sum_{N \in \mathbb{Z}} (\eta\left(\frac{x}{2^N}\right) + \eta\left(-\frac{x}{2^N}\right)) = 1, \quad \text{for } x \neq 0.$$

In the proof of Theorem 3 for the case $k = 2$ and $m = 1/4$ (extremal case) given in [48] Lemma 4 was combined with the inequality

$$\|D_\xi^{1/8} Q_N\left(\frac{e^{it\xi^3}}{(1+\xi^2)^{1/8}}\right)\|_{L_\xi^\infty l_N^1} < \infty,$$

to establish the main estimate in the proof.

For the study of persistence properties of the solution to the IVP associated to the k -gKdV equation (1.2) in exponential weighted spaces we refer to [40] and [15] and references therein.

Finally, we shall consider the BO equation (1.3).

4. THE BENJAMIN-ONO EQUATION (1.3)

The LWP in the Sobolev spaces $H^s(\mathbb{R})$ of the IVP associated to the BO equation (1.3) has been largely considered : in [1] and [31] LWP was established for $s > 3/2$, in [53] for $s \geq 3/2$, in [42] for $s > 5/4$, in [35] for $s > 9/8$, in [59] for $s \geq 1$, in [7] for $s > 1/4$, and in [30] LWP was proven in $H^s(\mathbb{R})$ for $s \geq 0$.

Real valued solutions of the IVP (1.3) satisfy infinitely many conservation laws (time invariant quantities), the first three are the following:

$$(4.1) \quad \begin{aligned} I_1(u) &= \int_{-\infty}^{\infty} u(x, t) dx, & I_2(u) &= \int_{-\infty}^{\infty} u^2(x, t) dx, \\ I_3(u) &= \int_{-\infty}^{\infty} (|D_x^{1/2} u|^2 - \frac{u^3}{3})(x, t) dx, \end{aligned}$$

where $D_x = \mathcal{H} \partial_x$.

The k -conservation law I_k provides an *a priori* estimate of the L^2 -norm of the derivatives of order $(k-2)/2$, $k > 2$ of the solution, i.e. $\|D_x^{(k-2)/2} u(t)\|_2$. This allows one to deduce GWP from LWP results.

In the BO equation the dispersive effect is described by a non-local operator and is significantly weaker than that exhibited by the Korteweg-de Vries (KdV) equation, i.e. $k = 1$ in (1.2). Indeed, it was proven in [46] that for any $s \in \mathbb{R}$ the map data-solution from $H^s(\mathbb{R})$ to $C([0, T] : H^s(\mathbb{R}))$ is not locally C^2 , and in [43] that it is not locally uniformly continuous. In particular, this implies that no LWP results can be obtained by an argument based only on a contraction method.

Consider the weighted Sobolev spaces

$$(4.2) \quad Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^{2r} dx), \quad \text{and} \quad \dot{Z}_{s,r} = \{f \in Z_{s,r} : \widehat{f}(0) = 0\} \quad s, r \in \mathbb{R}.$$

In [31] the following results were obtained:

- Theorem G.** (I) *The IVP associated to the BO equation (1.3) is GWP in $Z_{2,2}$.*
 (II) *If $\widehat{u}_0(0) = 0$, then the IVP associated to the BO equation (1.3) is GWP in $\dot{Z}_{3,3}$.*
 (III) *If $u(x, t)$ is a solution of the IVP associated to the BO equation (1.3) such that $u \in C([0, T] : Z_{4,4})$ for arbitrary $T > 0$, then $u(x, t) \equiv 0$.*

We observe that the linear part of the equation in (1.3) $L = \partial_t + \mathcal{H}\partial_x^2$ commutes with the operator $\Gamma = x - 2t\mathcal{H}\partial_x$, i.e.

$$[L; \Gamma] = L\Gamma - \Gamma L = 0.$$

Also, the solution $v(x, t)$ of the associated IVP

$$(4.3) \quad v(x, t) = U(t)v_0(x) = e^{-it\mathcal{H}\partial_x^2}v_0(x) = (e^{-it\xi|\xi|}\widehat{v}_0)^\vee(x),$$

satisfies that $v(\cdot, t) \in L^2(|x|^{2k} dx)$, $t \in [0, T]$, when $v_0 \in Z_{k,k}$, $k \in \mathbb{Z}^+$ for $k = 1, 2, \dots$ and

$$\int_{-\infty}^{\infty} x^j v_0(x) dx = 0, \quad j = 0, 1, \dots, k-3, \quad \text{if } k \geq 3.$$

In [32] the unique continuation result in $Z_{4,4}$ in Theorem G was improved:

Theorem I. *Let $u \in C([0, T] : H^2(\mathbb{R}))$ be a solution of the IVP (1.3). If there exist three different times $t_1, t_2, t_3 \in [0, T]$ such that*

$$(4.4) \quad u(\cdot, t_j) \in Z_{4,4}, \quad j = 1, 2, 3, \quad \text{then} \quad u(x, t) \equiv 0.$$

As in the previous cases, the goal was to extend the results in Theorem G and Theorem I from integer values to the continuum optimal range of indices (s, r) . In this direction one finds the following results established in [19]:

Theorem 4.

- (I) *Let $s \geq 1$, $r \in [0, s]$, and $r < 5/2$. If $u_0 \in Z_{s,r}$, then the solution $u(x, t)$ of the IVP associated to the BO equation (1.3) satisfies that $u \in C([0, \infty) : Z_{s,r})$.*
 (II) *For $s > 9/8$ ($s \geq 3/2$), $r \in [0, s]$, and $r < 5/2$ the IVP associated to the BO equation (1.3) is LWP (GWP resp.) in $Z_{s,r}$.*
 (III) *If $r \in [5/2, 7/2)$ and $r \leq s$, then the IVP associated to the BO equation (1.3) is GWP in $\dot{Z}_{s,r}$.*

Theorem 5. *Let $u \in C([0, T] : Z_{2,2})$ be a solution of the IVP associated to the BO equation (1.3). If there exist two different times $t_1, t_2 \in [0, T]$ such that*

$$(4.5) \quad u(\cdot, t_j) \in Z_{5/2, 5/2}, \quad j = 1, 2, \quad \text{then} \quad \widehat{u}_0(0) = 0, \quad (\text{so } u(\cdot, t) \in \dot{Z}_{5/2, 5/2}).$$

Theorem 6. *Let $u \in C([0, T] : \dot{Z}_{3,3})$ be a solution of the IVP (1.3). If there exist three different times $t_1, t_2, t_3 \in [0, T]$ such that*

$$(4.6) \quad u(\cdot, t_j) \in Z_{7/2, 7/2}, \quad j = 1, 2, 3, \quad \text{then} \quad u(x, t) \equiv 0.$$

Remarks : Theorem 5 and Theorem 6 show that the upper values of r for the persistence properties in $Z_{s,r}$ and $\dot{Z}_{s,k}$ in Theorem 4 are optimal. We recall that if $u \in C([0, T] : H^s(\mathbb{R}))$ is a solution of the BO equation (1.3) such that $\exists t_0 \in [0, T]$ for which $u(x, t_0) \in H^{s'}(\mathbb{R})$, $s' > s$, then $u \in C([0, T] : H^{s'}(\mathbb{R}))$. So it suffices to consider the most interesting case $s = r$ in (4.2).

The proof of Theorems 6 is based on weighted energy estimates and involves several inequalities concerning the Hilbert transform \mathcal{H} .

Among them one finds the A_p condition introduced in [47].

Definition 1. A non-negative function $w \in L^1_{loc}(\mathbb{R})$ satisfies the A_p inequality with $1 < p < \infty$ if

$$(4.7) \quad \sup_{Q \text{ interval}} \left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p-1} = c(w) < \infty,$$

where $1/p + 1/p' = 1$.

It was proven in [29] that this is a necessary and sufficient condition for the Hilbert transform \mathcal{H} to be bounded in $L^p(w(x)dx)$ (see [29],), i.e. $w \in A_p$, $1 < p < \infty$ if and only if

$$(4.8) \quad \left(\int_{-\infty}^{\infty} |\mathcal{H}f|^p w(x)dx \right)^{1/p} \leq c^* \left(\int_{-\infty}^{\infty} |f|^p w(x)dx \right)^{1/p},$$

In the case $p = 2$, a previous characterization of w in (4.7) was found in [28]. However, even though the main case is for $p = 2$, the characterization (4.7) will be the one used in the proof. In particular, one has that in \mathbb{R}

$$(4.9) \quad |x|^\alpha \in A_p \Leftrightarrow \alpha \in (-1, p-1).$$

In order to justify some of the arguments in the proofs one need some further continuity properties of the Hilbert transform. More precisely, the proof requires the constant c^* in (4.8) to depend only on $c(w)$ the constant describing the A_p condition (see (4.7)) and on p . In [52] precise bounds for the constant c^* in (4.7) were given which are sharp in the case $p = 2$ and sufficient for the purpose in [19].

It will be essential in the arguments in [19] that some commutator operators involving the Hilbert transform \mathcal{H} are of “order zero”. More precisely, one shall use the following estimate: $\forall p \in (1, \infty)$, $l, m \in \mathbb{Z}^+ \cup \{0\}$, $l+m \geq 1$ $\exists c = c(p; l; m) > 0$ such that

$$(4.10) \quad \|\partial_x^l [\mathcal{H}; a] \partial_x^m f\|_p \leq c \|\partial_x^{l+m} a\|_\infty \|f\|_p.$$

In the case $l+m = 1$, (4.10) is Calderón’s first commutator estimate [8]. The case $l+m \geq 2$ of the estimate (4.10) was proved in [14].

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